TIF 150: Information theory for complex systems

Time: March 17, 2016, afternoon.
Allowed material: Calculator (type approved according to Chalmers rules). "Beta".
Teacher and examiner: Kristian Lindgren (031-7723131)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on April 7.

1. Balance measurements

You have 5 balls, and you know that 4 are of normal weight, while 1 is slightly heavier or lighter, but you do not know which one is deviating and how. In addition, you have separately four extra balls that you know are all of normal weight. Using as few balance measurements as possible (considering the worst case), you are expected to identify the deviating ball, but it is not necessary to tell whether it is heavier or lighter. You may use the extra balls in the measurements.

(a) According to information theory, what is the initial entropy of the 5-ball system? How much information would you get from ideal measurements using a balance (an ideal measurement is one that provides the most expected information)? Can this provide information about how many measurements you will need in order to solve the problem?

(b) Find a sequence of measurements that identifies the deviating ball. Determine the remaining uncertainty (entropy) after the successive measurement steps.

(10 p)

2. An equilibrium spin system

Consider an infinite one-dimensional spin system in which each lattice cite (cell) can be in one of four possible states: (↑, 0), (↓, 0), (↑, 1), and (↓, 1). For example:

↑	↓	↓	↑	↑	↓	↓	↓	↑	↑	↑	↑	↓
0	1	0	1	0	1	0	1	0	1	0	1	0

Each local state thus has two components: A spin component, ↓ or ↑, which determines the contribution the energy density, so that neighbouring parallel spins contribute with energy –J and antiparallel spins with +J. The second component, 0 or 1, does not contribute to the energy density, but there is a strict constraint requiring that a 0 must have 1's as neighbours, and vice versa, as indicated in the above example.

(a) If the average energy per cell is u, what are the equilibrium characteristics of the system? Or, in other words, what are the probabilities over sequences of symbols that characterise the system? **You need not to solve the equations but you should set up the equations that determine the solution.**

(6 p)

(b) Consider a cellular automaton rule that transforms a state of the spin system (i.e., the whole infinite sequence) as follows. The rule depends on the two local components and the spin component of the neighbours: If the second component is a 0, the local spin state does not change. If the second component is a 1, the local spin state will flip if the neighbouring spin states are one of each: ↓ and ↑ (but the order is irrelevant), i.e., there is a spin flip if the local energy does not change. In all situations, the second component will switch, i.e., 0 to 1 and vice versa. Does this cellular automaton rule change the entropy density? (Of course, you need to provide an argument for your answer.)

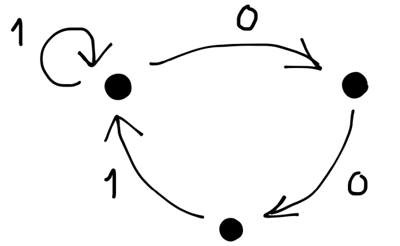
(4 p)

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule R18, i.e., neighbourhoods 100 and 001 map to 1, and all others to 0.

(a) Is this rule almost reversible or irreversible?

Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node they have equal probabilities).



(b) What is the initial entropy density (at $t = 0$)? (Entropy density here refers to the entropy per symbol in the CA state, or equivalently, the entropy rate of the stochastic process that generates the CA state.)

(c) Derive the finite state automaton of the CA state after one time step ($t = 1$).

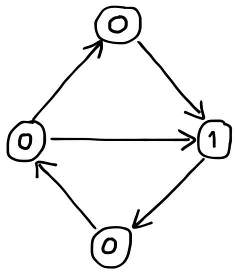
(d) What is the entropy at this time ($t = 1$)?

(e) What is the entropy after two time steps ($t = 2$)?

(10 p)

4. Correlation complexity.

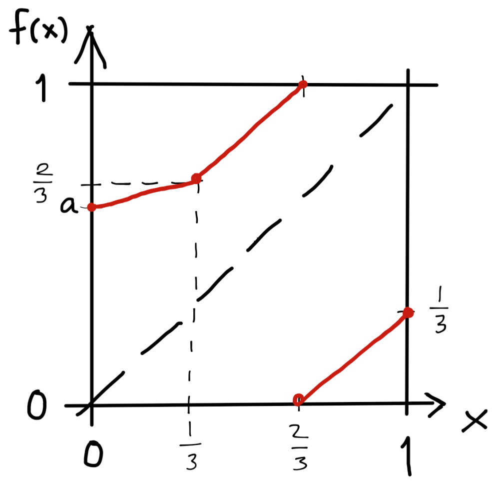
Consider the process defined by the finite state automaton below. When two arcs leave a node they have the same probability.



Determine the correlation complexity η.

(10 p)

5. Chaos and information. Let a piecewise linear map $f(x)$ be defined by the figure below, where $0 \le a \le 2/3$, and where the mapping is determined by $f(0) = a$, $f(1/3) = 2/3$, $f(2/3) = 1$, $f(2/3^+) = 0$, and $f(1) = 1/3$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \ .$$

(a) Start with a at 2/3 and discuss how the dynamics changes when a is decreased. Determine whether there is a stable fixed point, stable periodic orbit, or chaos. Is there a critical value for a, for which there is a change in dynamical characteristics?

(b) Suppose now that $a = 0$:

- Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ.
- Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

(10 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu) = S[P] + \sum_{k=1}^{r}\lambda_k\left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu-1)\left(1-\sum_{i=1}^{n} p_i\right),$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r}\lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n}\exp(-\sum_k \lambda_k f_k(j)), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty}\sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1})\sum_{x_n} p(x_n \mid x_1\ldots x_{n-1})\log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}(S_n - S_{n-1}) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n}S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$k_m = \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1}) K[P^{(0)};P] = \tag{3.16}$$

$$= \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1}) \sum_{x_m} p(x_m|x_1x_2\ldots x_{m-1}) \log\frac{p(x_m|x_1x_2\ldots x_{m-1})}{p(x_m|x_2\ldots x_{m-1})},$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chaos and information

$$\int dx \mu(x) \varphi(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \varphi(f^k(x(0))),$$

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln\left|f'(x(k))\right| = \int dx \mu(x) \ln\left|f'(x)\right|$$

$$h(\mu, \mathbf{A}) = \lim_{n \to \infty} \frac{1}{n} H(B^{(n)}) = \lim_{n \to \infty} \left(H(B^{(n+1)}) - H(B^{(n)}) \right), \quad s_\mu = \lim_{diam(\mathbf{A}) \to 0} h(\mu, \mathbf{A}), \quad s_\mu = \lambda$$