5.1

First, note that we know the whole system, so we can simply enumerate all configurations and maximize the entropy of the prob. distribution.

Configuration	$\begin{matrix}\uparrow\uparrow\\\uparrow\uparrow\end{matrix}$	$\begin{matrix}\uparrow\downarrow\\\uparrow\uparrow\end{matrix}$	$\begin{matrix}\uparrow\uparrow\\\downarrow\downarrow\end{matrix}$	$\begin{matrix}\uparrow\downarrow\\\downarrow\uparrow\end{matrix}$	
Multiplicity	2	8	4	2	sum $16 = 2^4$ Good!
Energy h_i	$4J$	0	0	$-4J$	
Numbers (i)	1, 2	3–10	11–14	15, 16	

Now maximize $S[p] = \sum_i p_i \ln \frac{1}{p_i}$

subject to $\begin{cases} u = \sum_i h_i p_i, \\ 1 = \sum_i p_i . \end{cases}$

Equivalent to maximizing the unconstrained Lagrangian

$$L = \sum_i p_i \ln \frac{1}{p_i} + \beta\left(u - \sum_i h_i p_i\right) + \lambda\left(1 - \sum_i p_i\right).$$

$$0 = \frac{\partial L}{\partial p_i} = -1 - p_i - \beta h_i - \lambda$$

$$\Rightarrow \quad p_i = e^{-\mu - \beta h_i}, \quad \text{where } \mu = 1 + \lambda.$$

5.1 cont.

Now note that p_i depends only on β, μ, h_i. All configurations with equal energy must also have equal probability: $p_1 = p_2$; $p_3 = p_4 = \cdots = p_{14}$; $p_{15} = p_{16}$.

Also note that $\frac{p_1}{p_{15}} = e^{-8J\beta}$, $\frac{p_3}{p_{15}} = e^{-4J\beta}$.

$$\Rightarrow 0 = \frac{\partial L}{\partial \mu} = 1 - \sum_i p_i = 1 - p_{15}\left(2e^{-8J\beta} + 12e^{-4J\beta} + 2\right)$$

$$\Rightarrow \begin{cases} p_{15} = \dfrac{1}{2\left(e^{-8J\beta} + 6e^{-4J\beta} + 1\right)}, \\ p_1 = p_2 = p_{15}\, e^{-8J\beta}, \\ p_3 = \cdots = p_{14} = p_{15}\, e^{-4J\beta}. \end{cases}$$

(And $\beta = \frac{1}{k_B T}$ if you want to express the solution in temperature.)

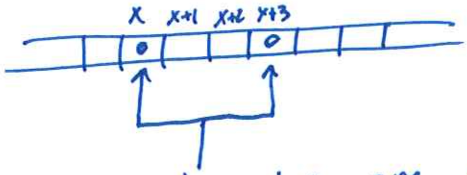
Note. As $T \to 0_+$, $\beta \to +\infty$ and

$p_{15} = p_{16} \to \frac{1}{2}$.

As $T \to \infty$, $\beta \to 0$ and

$p_i \to \frac{1}{16}$.

5.2



these two are correlated.

We need to maximize ΔS_4.

Configuration	$\uparrow \cdot \cdot \uparrow$	$\uparrow \cdot \cdot \downarrow$	$\downarrow \cdot \cdot \uparrow$	$\downarrow \cdot \cdot \downarrow$
Multiplicity	4	4	4	4
Energy	$-J$	$+J$	$+J$	$-J$
Probability	p_0	p_1	p_1	p_2

(p_1 must be the same number in an infinite sequence)

This is essentially three shifted / overlaid one-dimensional Ising models as in the lecture notes. Stationary solution must be the same, but scaled by $\frac{1}{4}$:

$$p_0 = p_2 = \frac{1}{8(1+e^{-2\beta J})}, \quad p_1 = \frac{1}{8(1+e^{2\beta J})}$$

5.3

Maximize $s \Leftrightarrow$ maximize $\lim_{m\to\infty} \Delta S_m \Longleftrightarrow$ maximize ΔS_2, since we only need correlations in blocks of length 2.

So maximize $S_2 - S_1$.

Conf.	$\uparrow\uparrow$	$\uparrow\rightarrow$	$\uparrow\downarrow$
Multiplicity n_i	4	8	4
Energy h_i	$-J$	0	J
Probability	p_1	p_2	p_3

$(\Sigma = 16)$

Note that if all symmetries are preserved all "rotations" must be equally likely, so then we only have three distinct probabilities.

And due to symmetry all 1-blocks $(\uparrow, \rightarrow, \downarrow, \leftarrow)$ are equally likely, so $S_1 = \log 4$ unavoidably.

Thus, max ΔS_2

subject to

$$\begin{cases} u = \sum_i p_i n_i h_i \\ 1 = \sum_i p_i n_i \end{cases}$$

Is equivalent to unconstrained maximization of

$$L = \sum_i p_i n_i \ln \frac{1}{p_i} + \beta\left(u - \sum_i p_i n_i h_i\right) + (\mu - 1)\left(1 - \sum_i p_i n_i\right).$$

5.3 cont.

$$p_i = e^{-\mu-\beta h_i} \qquad \left(\text{since } 0 = \frac{\partial L}{\partial p_i}\right)$$

$$\frac{p_1}{p_3} = e^{2J\beta}, \quad \frac{p_2}{p_3} = e^{J\beta}$$

$$\Rightarrow \quad 0 = \frac{\partial L}{\partial \mu} = 1 - \sum p_i n_i = 1 - 4\left(e^{2J\beta} + 2e^{J\beta} + 1\right) p_3$$

$$\Rightarrow \quad p_3 = \frac{1}{4\left(e^{2J\beta} + 2e^{J\beta} + 1\right)}$$

As $T \to 0$, $p_3, p_2 \to 0$ and $p_1 \to \frac{1}{4}$.

As $T \to \infty$, $p_1, p_2, p_3 \to \frac{1}{16}$.

When $T \to 0$, $S_2 \to \log 4 \Rightarrow s \to 0$.

Physical interpretation: we tend towards solutions like

$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \ldots$

$\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \ldots$

5.6

configuration	AA CC	BB	AC CA	BC CB AB BA
multiplicity n_i	2	1	2	4
energy h_i	$+J$	$+J$	$+J$	$-J$
probability	p_1	p_2	p_3	p_4

$\sum_i n_i p_i = 2p_1 + p_2 + 2p_3 + 4p_4 = 1$ (prob. normalized)

$u = \sum_i n_i p_i h_i$ (energy constraint)

single symbol probabilities:

$q_1 = P(A) = p_1 + p_3 + p_4$

$q_2 = P(B) = p_2 + 2p_4$

$q_3 = P(C) = p_1 + p_3 + p_4$

Check $\sum_j q_j = \sum_i n_i p_i$ OK!

Now maximize $s = S_2 - S_1 = \sum_i n_i p_i \ln \frac{1}{p_i} - \sum_j q_j \ln \frac{1}{q_j}$

subject to

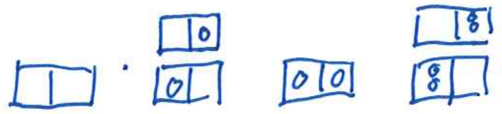
$1 = \sum_i n_i p_i$

$u = \sum_i n_i p_i h_i$ (using Lagrangian optimization)

5.8.

How long blocks do we need to consider?
2, because it is sufficient to require "only blank sites <u>before</u> and <u>after</u> married couples".

Configuration



Mult. n_i	1	2	1	2
Prob.	p_1	p_2	p_3	p_4

Single symbol probabilities:

$$q_1 = p(\square) = p_1 + p_2 + p_4$$
$$q_2 = p(\boxed{0}) = p_2 + p_3$$
$$q_3 = p(\boxed{8}) = p_4$$

Sum $\sum_j q_j = p_1 + 2p_2 + p_3 + 2p_2$ OK!

Now maximize $\Delta S_2 = \sum_{i=1}^{4} p_i n_i \ln \frac{1}{p_i} - \sum_{j=1}^{3} q_j \ln \frac{1}{q_j}$

subject to

$$1 = \sum_i p_i n_i$$
$$\rho = \frac{p(\boxed{0}) + 2p(\boxed{8})}{\sum p_i n_i}$$