TIF 150: Information theory for complex systems

Time: March 16, 2018, afternoon.
Allowed material: Calculator (type approved according to Chalmers rules). “Beta”.
Teacher and examiner: Kristian Lindgren (031-7723131)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on April 6.

1. Balance measurements

You have 7 balls, and you know that 5 are of normal weight, while 2 are slightly heavier (having the same weight deviation), but you do not know which ones are deviating. Using as few balance measurements as possible (considering the worst case), you are expected to identify the two deviating balls.

(a) According to information theory, what is the initial entropy of the 7-ball system? How much information would you get from ideal measurements using a balance (an ideal measurement is one that provides the most expected information)? What information does this provide about how many measurements you will need in order to solve the problem?

(b) Find a sequence of measurements that identifies the deviating balls. Determine the remaining uncertainty (entropy) after the successive measurement steps. You only need to consider the worst outcome (or one of them if there are several) in each measurement, i.e., the most probable measurement result.

(8 p)

2. Maximum entropy description of a one-dimensional world

You are studying the population density patterns in an infinite one-dimensional lattice world. By studying the population patterns in this world, you have found the following regularities: Each site in the lattice either has a high population density ρ_H, a low population density ρ_L, or is empty (has no population). Thus, each site can be classified according to its population density as being one of the types $\{H, L, 0\}$. The population lives in two types of settlements: cities and villages. Cities are regions of one or more H sites, neighboured by exactly one L site on each side. Villages consist of a single L site. Between any two settlements, there is always at least one empty site. You have also come across a population survey that says that the average population density in the world is ρ.

For example, a portion of the world could look like this:

...	0	0	L	H	H	H	L	0	L	0	L	H	L	0	0	0	L	0	...

(a) Use the maximum entropy formalism to develop a statistical description of this world that has maximal entropy and is consistent with the regularities you have observed. You do not need to solve the problem fully, but only write down equations that can be solved to find the probability distribution over different symbol sequences.

(6 p)

(b) What fraction of the population lives in the cities? You may express your answer in terms of ρ, ρ_H, ρ_L, and any of the results you would find from solving the equations in part (a).

(3 p)

(c) Draw a finite state automaton that could be used to generate random instances of this lattice world consistent with the maximum entropy solution from part (a). You do not have to work out the transition probabilities between nodes, but indicate which transitions are possible by connecting two nodes with an arrow if and only if the transition probability is nonzero.

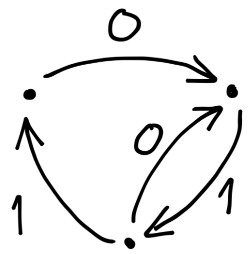
(3 p)

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule R37, i.e., neighbourhoods $101, 010$, and 000 map to 1, and all others to 0.

(a) Is this rule reversible, almost reversible, or irreversible? **(1 p)**

Let the initial state be characterized by the following finite state automaton (where it is assumed that if two arcs leave the same node they have equal probabilities)



(b) What is the initial entropy (at $t = 0$)? **(2 p)**

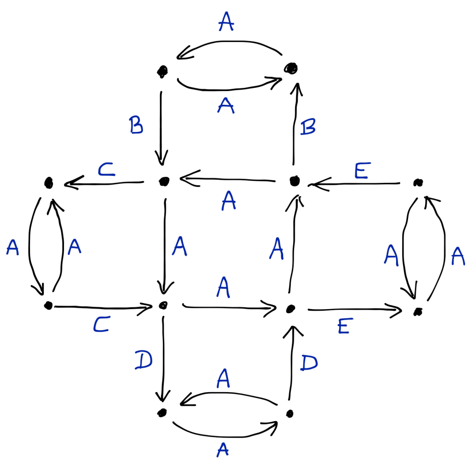
(c) Derive the finite state automaton of the CA state after one time step ($t = 1$). **(2 p)**

(d) What is the entropy at this time ($t = 1$)? **(1 p)**

(e) What is the correlation complexity η at $t = 0$ and at $t = 1$, respectively? **(8 p)**

4. Correlation complexity.

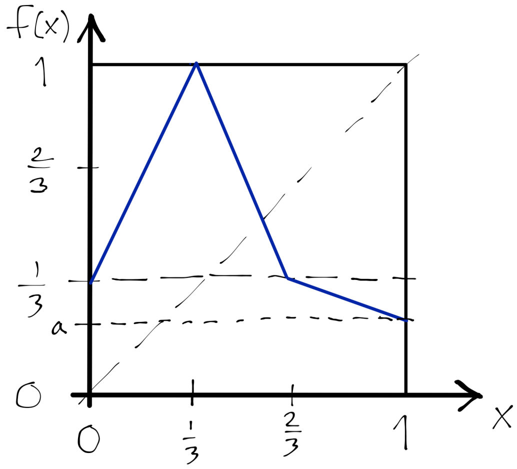
Consider the process generating sequences of symbols from the alphabet {A, B, C, D, E} defined by the finite state automaton below. When two arcs leave a node they have the same probability (i.e., 1/2 each).



Determine the correlation complexity η.

(6 p)

5. Chaos and information. Let a piecewise linear map $f(x)$ be defined by the figure below, where $0 \le a \le 1/3$, and where the map is determined by $f(0) = 1/3$, $f(1/3) = 1$, $f(2/3) = 1/3$, and $f(1) = a$.



Consider the dynamical system

$$x_{t+1} = f(x_t) \ .$$

(a) Start with a at 1/3 and discuss how the dynamics changes when a is decreased. Determine whether there is a stable fixed point, stable periodic orbit, or chaos. Is there a critical value for a, for which there is a change in dynamical characteristics?

(b) Suppose now that $a = 0$:

- Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ.
- Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

(10 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k + 1$ constraints) using the Lagrangian L

$$L(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu) = S[P] + \sum_{k=1}^{r} \lambda_k\left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu - 1)\left(1 - \sum_{i=1}^{n} p_i\right),$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r} \lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n} \exp(-\sum_k \lambda_k f_k(j)), \quad \frac{\partial \mu(\lambda)}{\partial \lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty} \sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1}) \sum_{x_n} p(x_n \mid x_1\ldots x_{n-1}) \log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}(S_n - S_{n-1}) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n} S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$k_m = \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1}) K[P^{(0)};P] = \tag{3.16}$$

$$= \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1}) \sum_{x_m} p(x_m|x_1x_2\ldots x_{m-1}) \log\frac{p(x_m|x_1x_2\ldots x_{m-1})}{p(x_m|x_2\ldots x_{m-1})},$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty} k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\, p(x-w) \quad , \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2 + r^2}}\exp\left(-\frac{x^2}{2(b^2 + r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chaos and information

$$\int dx\mu(x)\varphi(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}\varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\ln\left|f'(x(k))\right| = \int dx\mu(x)\ln\left|f'(x)\right|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty}\frac{1}{n}H(B^{(n)}) = \lim_{n\to\infty}\left(H(B^{(n+1)}) - H(B^{(n)})\right), \quad s_\mu = \lim_{diam(\mathbf{A})\to 0} h(\mu,\mathbf{A}), \quad s_\mu = \lambda$$