The Ising model for magnetism

The Ising model (Ernst Ising, 1900-1998; PhD thesis, 1924)

- Sequence of atoms with magnetic moments (spins), $\uparrow$ or $\downarrow$.
- Energy contributed from pairwise interaction:
  $\uparrow\uparrow$ gives $-J$, $\downarrow\uparrow$ gives $+J$, etc.
- Intuition: low temperature $\leftrightarrow$ low energy, and spins point in same direction leading to magnetisation.
- Turns out not to work in that way in one dimension!

Context and questions

We investigate the approach towards equilibrium in a dynamic Ising model with microscopic reversibility and conserved energy.

- It what sense can it be said the equilibrium is approached?
- Microscopic reversibility implies conserved entropy; how can that be consistent with an approach towards equilibrium?
- Can an information-theoretic characterisation, e.g., identifying information in correlations provide a description of the process?

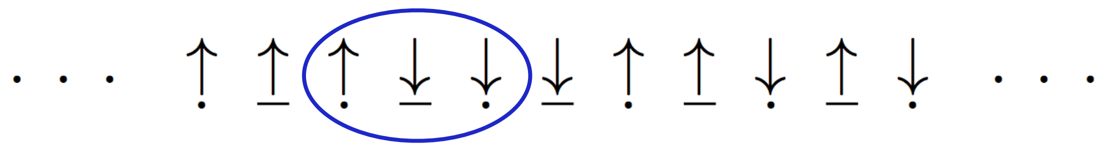
Lindgren & Olbrich (2017): The approach towards equilibrium in a reversible Ising dynamics model — an information-theoretic analysis based on an exact solution

A reversible and energy conserving Ising dynamics

The Q2R rule[1] in one dimension:

- Update spins by alternating between the two sub-lattices, of odd and even positions respectively.
- Interaction energy h given by: $h(\uparrow\uparrow) = -1$, $h(\uparrow\downarrow) = +1$.
- A spin in an updating state is flipped if energy is not changed.

t



Updating state: _

Quiescent state: .

[1] Vichniac, 1984

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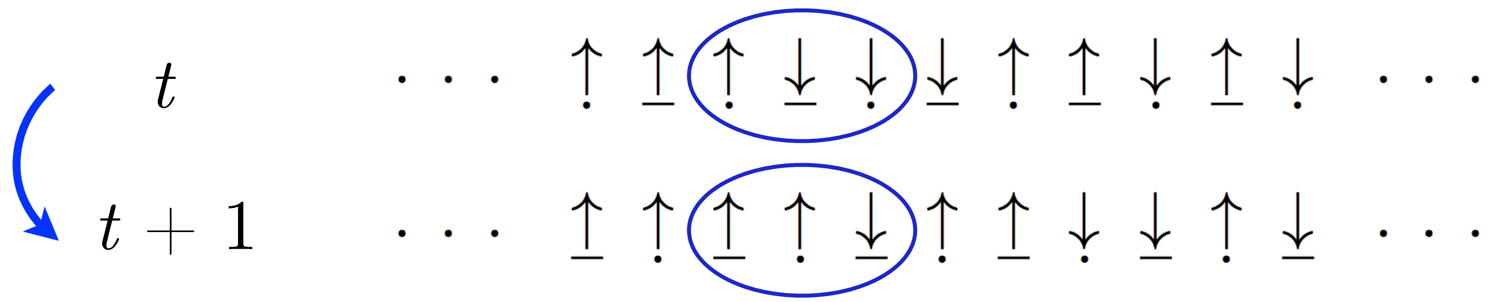

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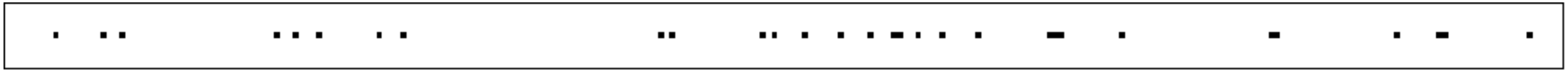
Reversible Ising dynamics in 1D



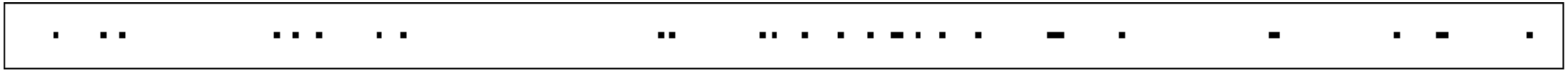
Showing 250 sites from a system of length 2^{14}.

The entropy per site s is lower than for the corresponding equilibrium state for this average energy. The entropy calculated from the correlations present is conserved in the time dynamics.

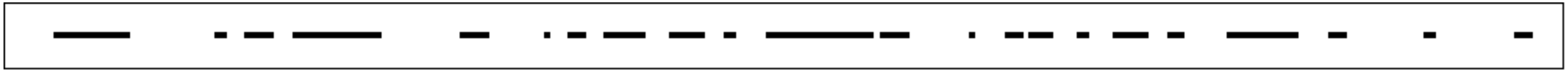
How is this conserved entropy consistent with a process that seemingly brings the system to equilibrium?

Reversible Ising dynamics in 1D

Initial state:



State at t=300:



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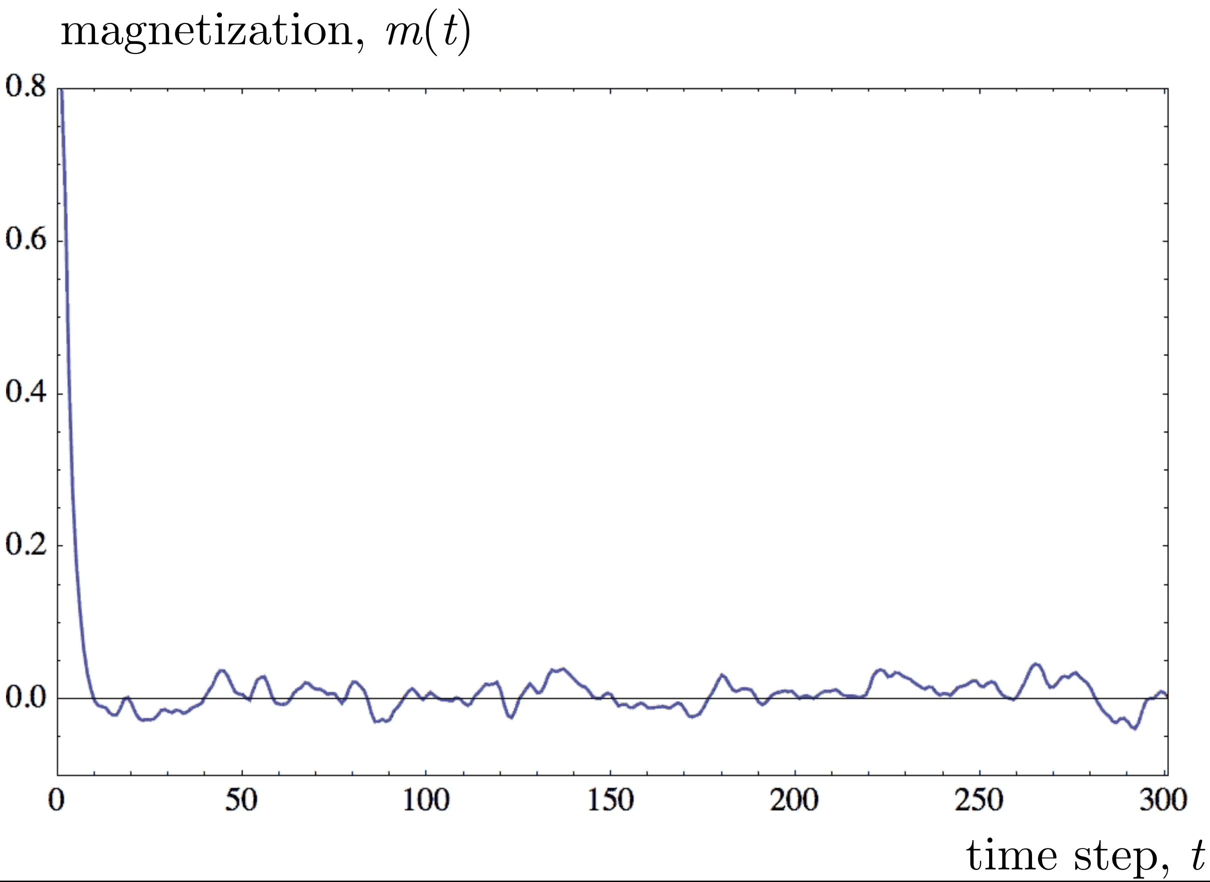
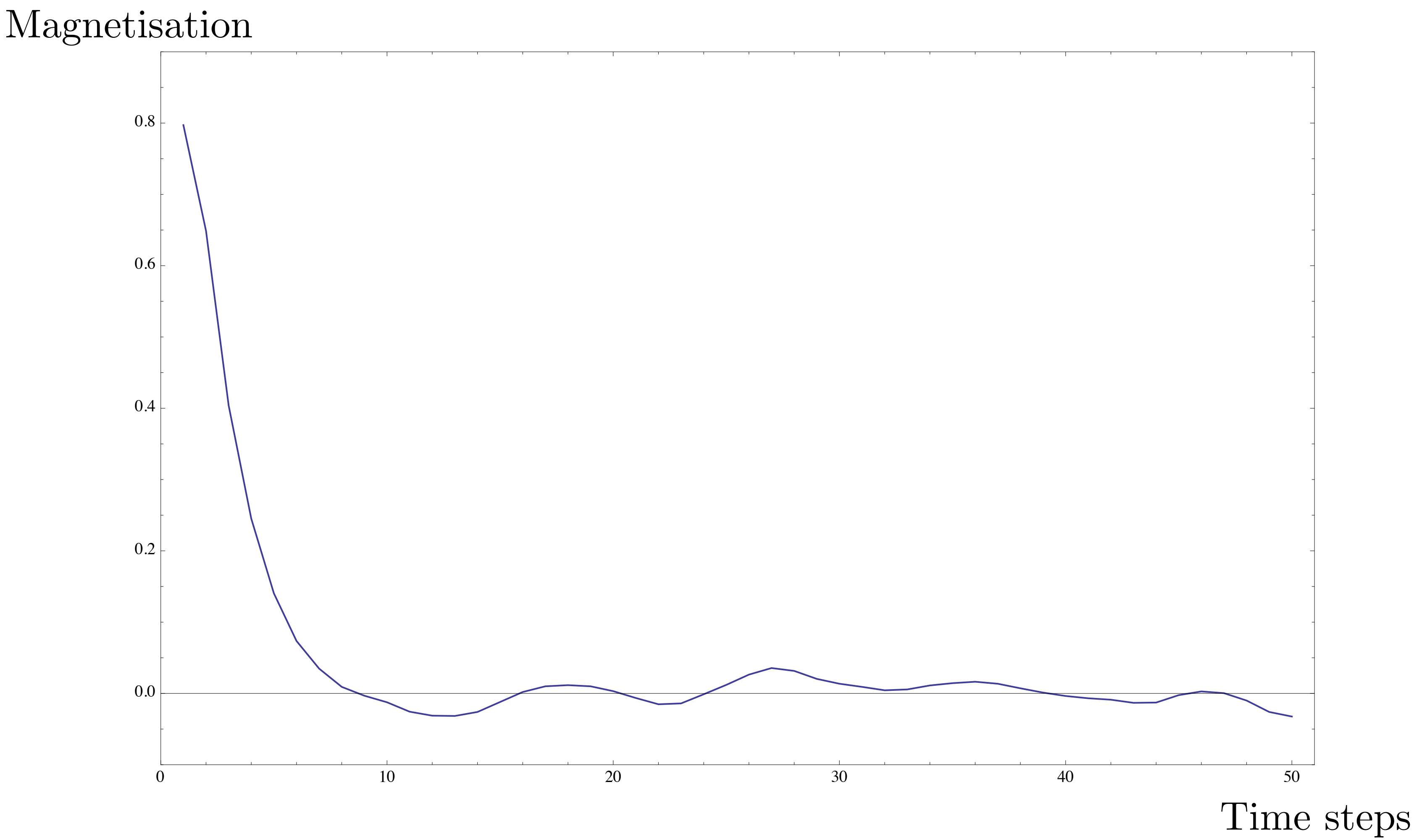


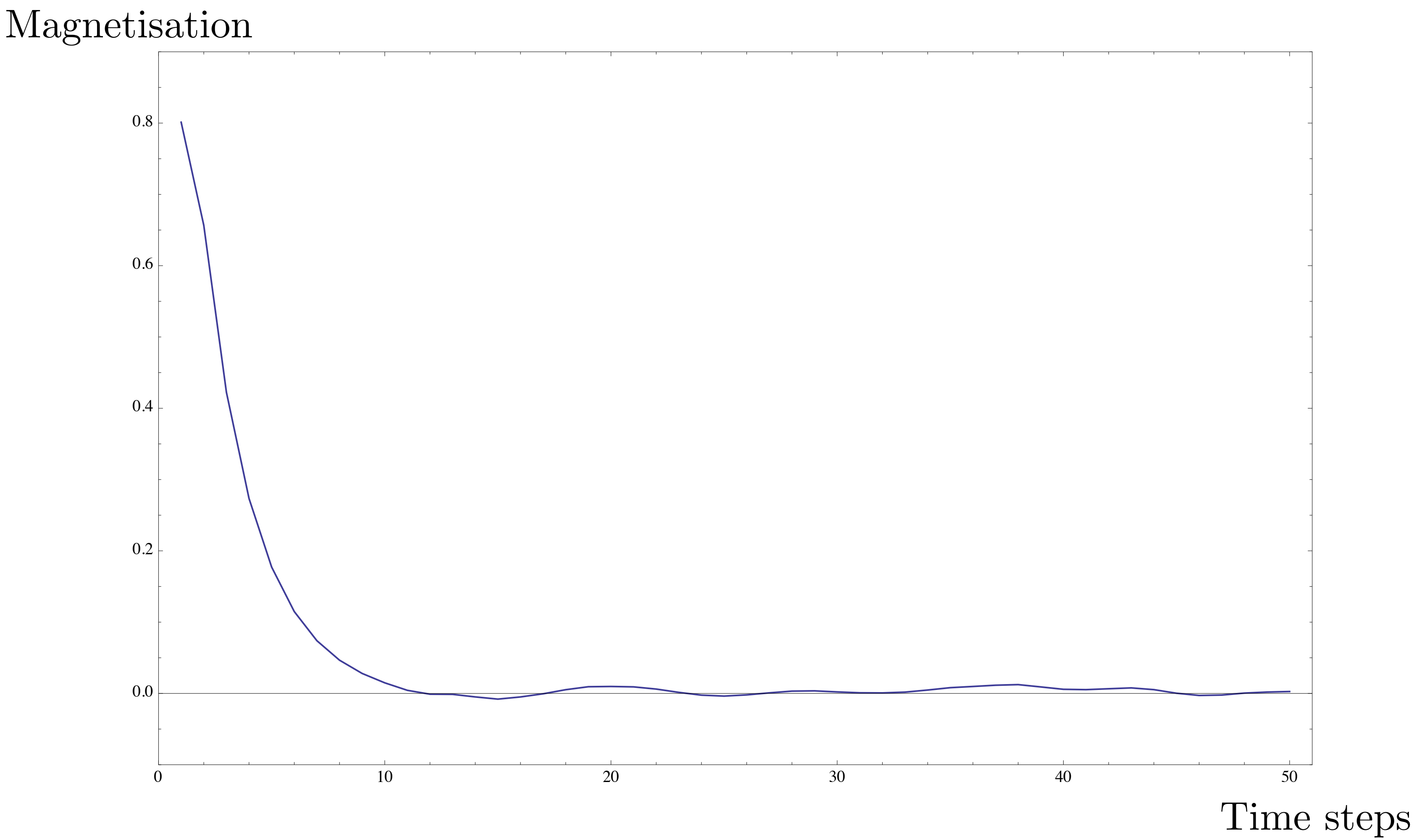
Approach to zero magnetisation

System size: 10^4 spins



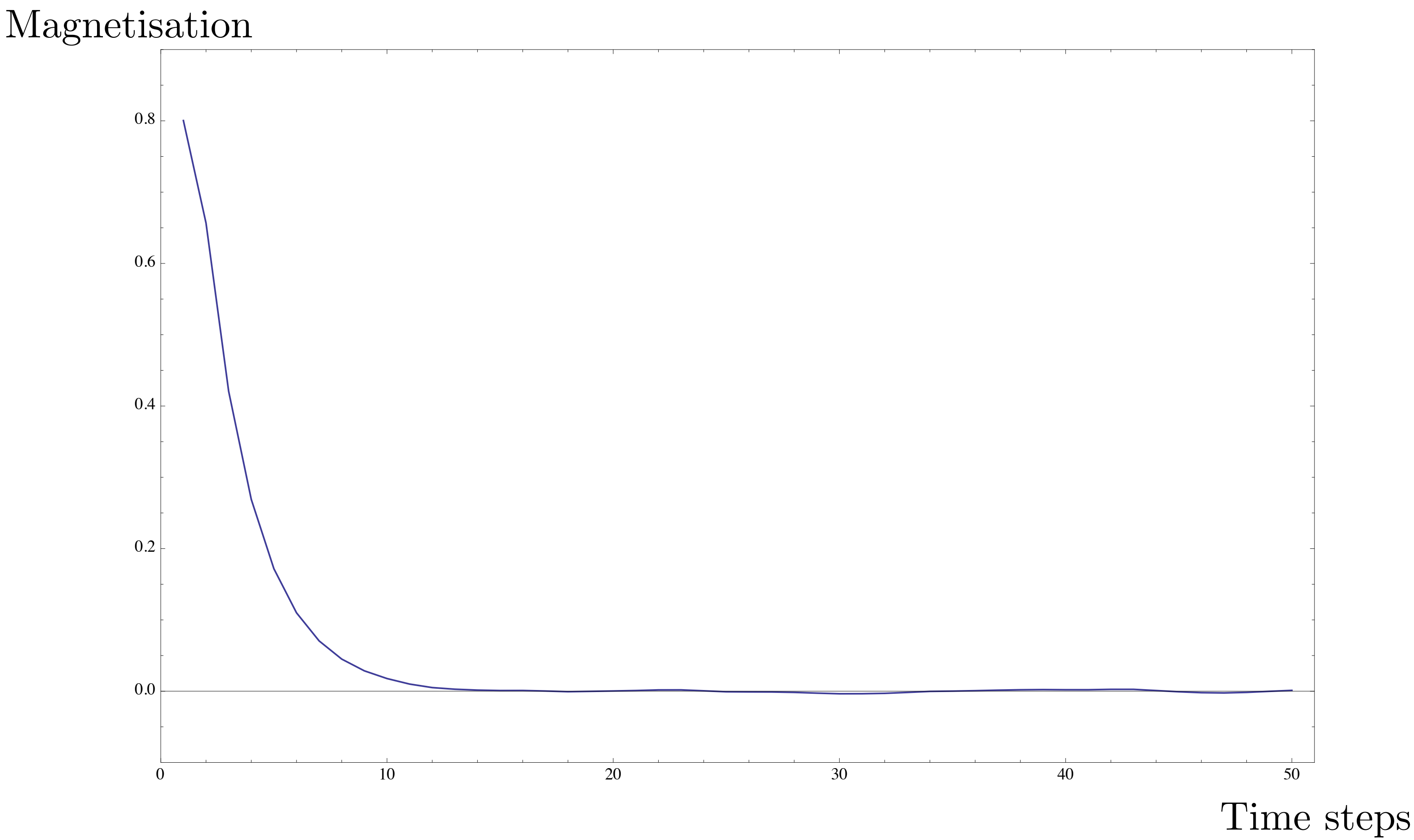
Approach to zero magnetisation

System size: 10^5 spins



Approach to zero magnetisation

System size: 10^6 spins



Mathematical representation of the dynamics

Quiescent states: $\underset{\bullet}{0}$ $\underset{\bullet}{1}$

Updating states: $\underline{0}$ $\underline{1}$

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Rule table (omitting symmetric entries):

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Note: The rule is addition (mod 2) for the updating sites

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Proposition: Updating and quiescent states, $\underline{s}_{i,t}$ and $\underset{\cdot}{s}_{j,t}$, respectively, at time t>0, can be expressed as sums of initial spin states ξ_j over certain intervals,

$$\underset{\cdot}{s}_{j,t} = \xi_{j-t} + \xi_{j-t+1} + \ldots + \xi_j + \ldots + \xi_{j+t-1} + \xi_{j+t} \pmod 2$$
$$\underline{s}_{i,t} = \xi_{i-t+1} + \ldots + \xi_i + \ldots + \xi_{i+t-1} \pmod 2$$

Magnetization — formal results

Proposition:

The magnetization $m(t) = n(\uparrow,\ t) - n(\downarrow,\ t)$ decays exponentially with t :

$$m(t) = \frac{1}{2}(1-2p)^{2t-1} + \frac{1}{2}(1-2p)^{2t+1}$$

where p is the initial frequency of $\downarrow$. This explains the fast relaxation.

For $p = 0.1$, we have at time $t = 10$: $m(10) = 0.012...$

Approach to zero magnetisation

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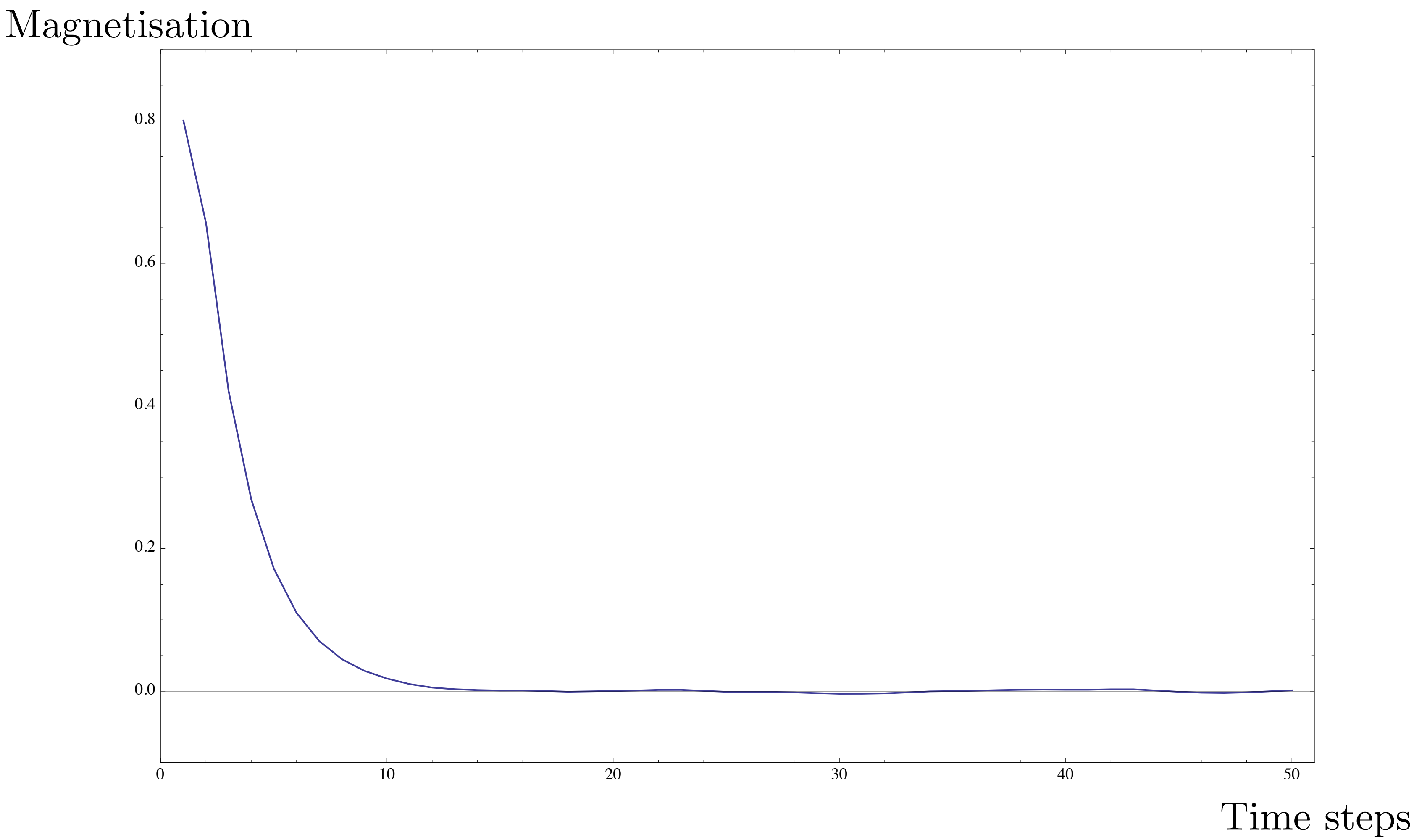
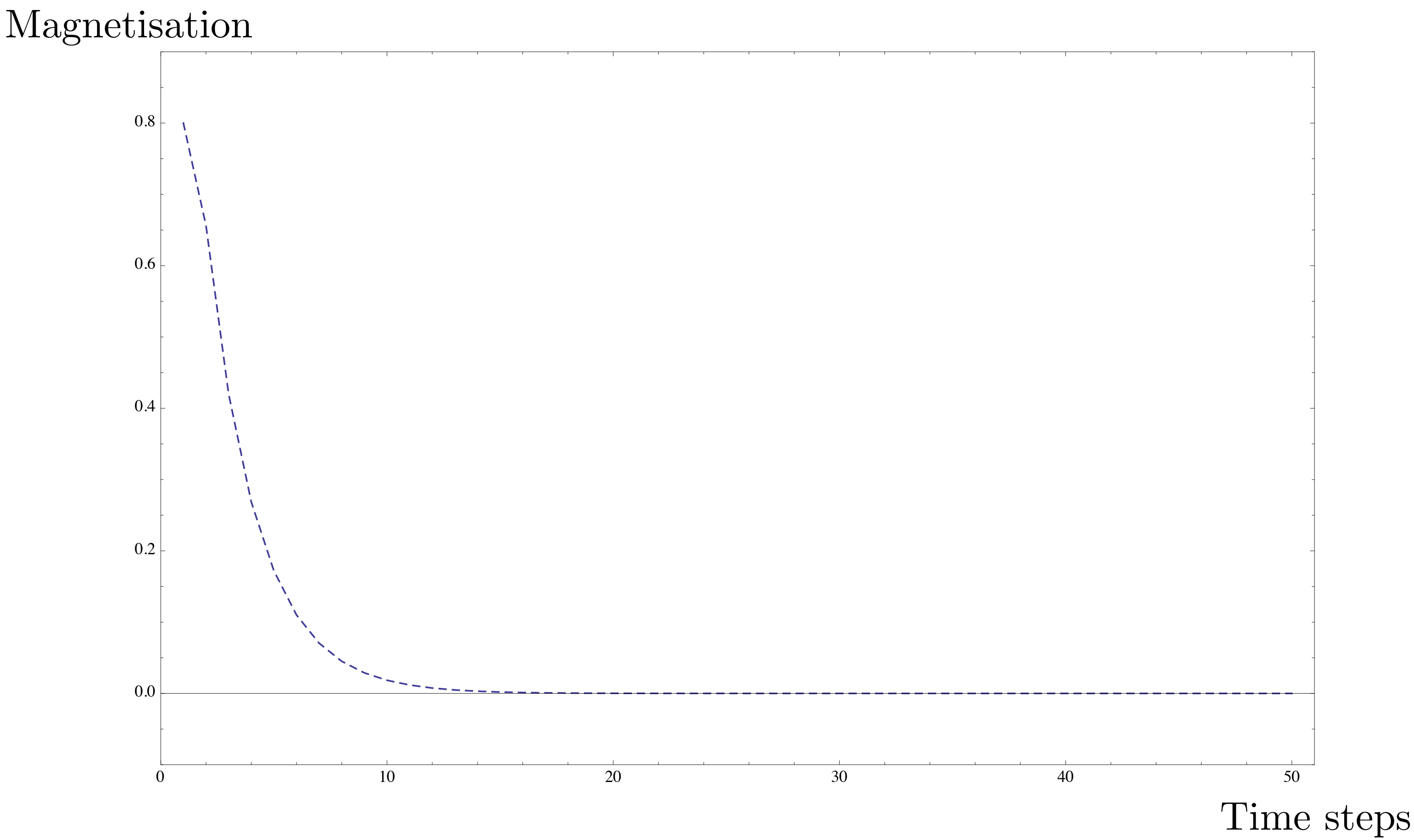


Approach to zero magnetisation

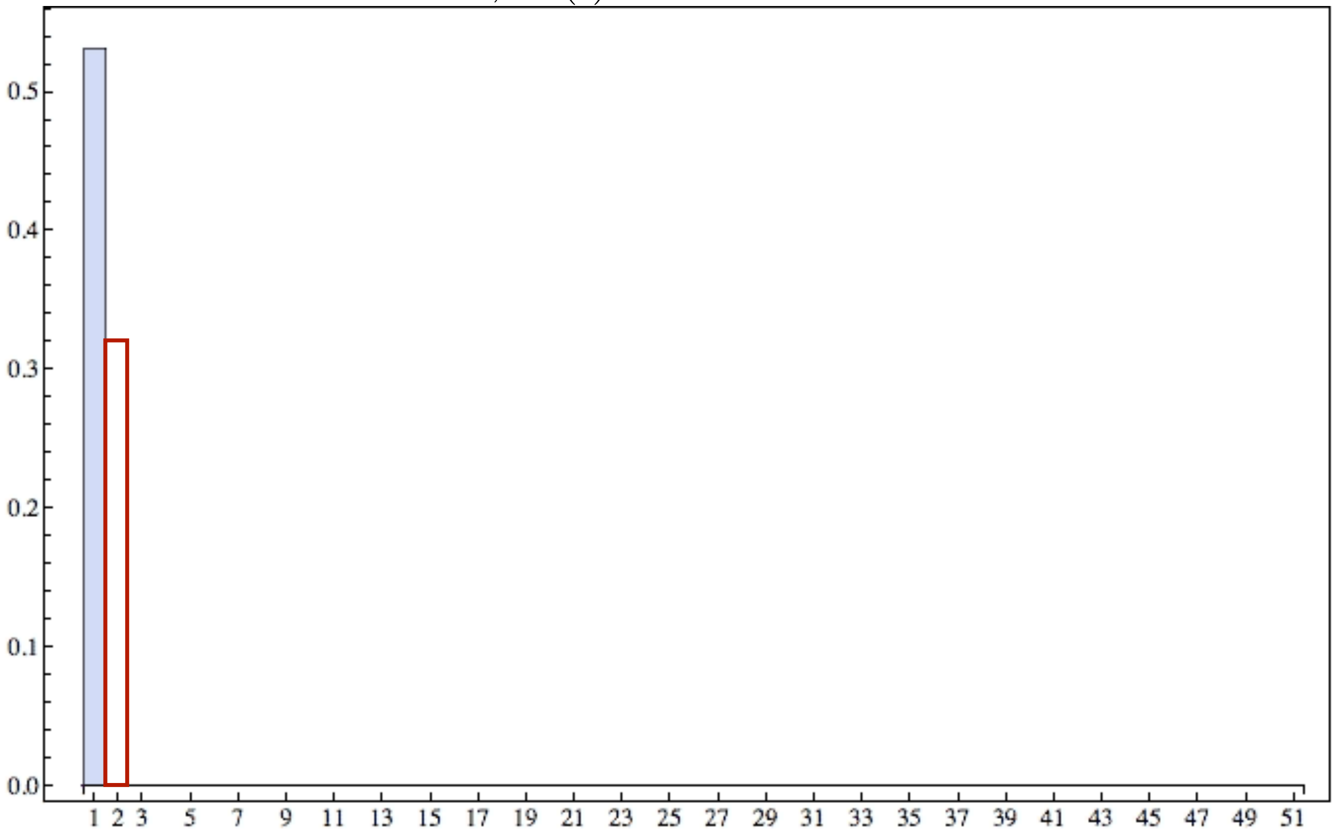
Analytic solution: infinite system



Evolution of correlation information

Showing the correlation information k_m, including the density information k_1, up length $m = 51$, over time ($t = 0, 1, 2, \ldots, 60$).

Contributions to the correlation information



— correlation information (k_2) at equilibrium

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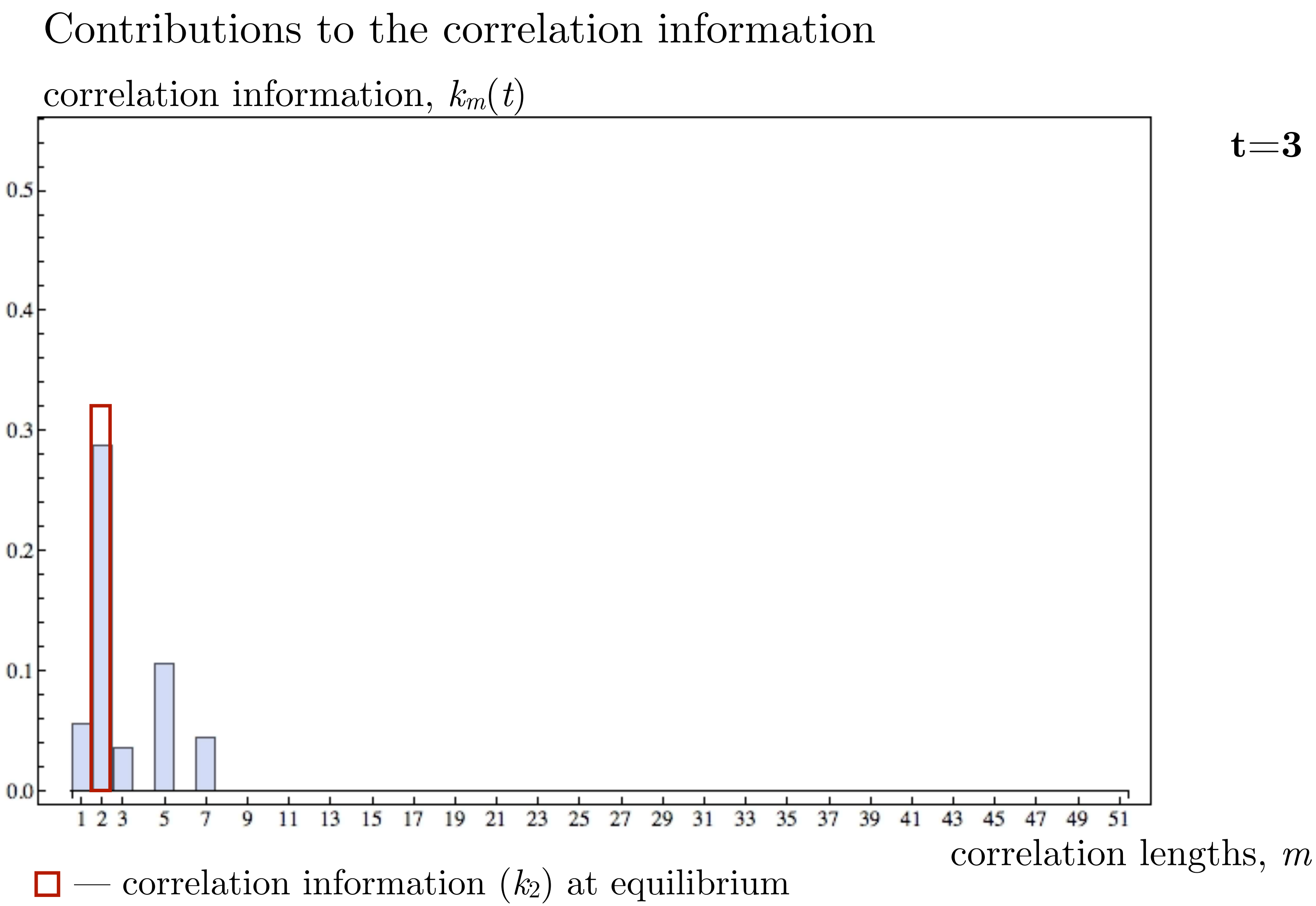

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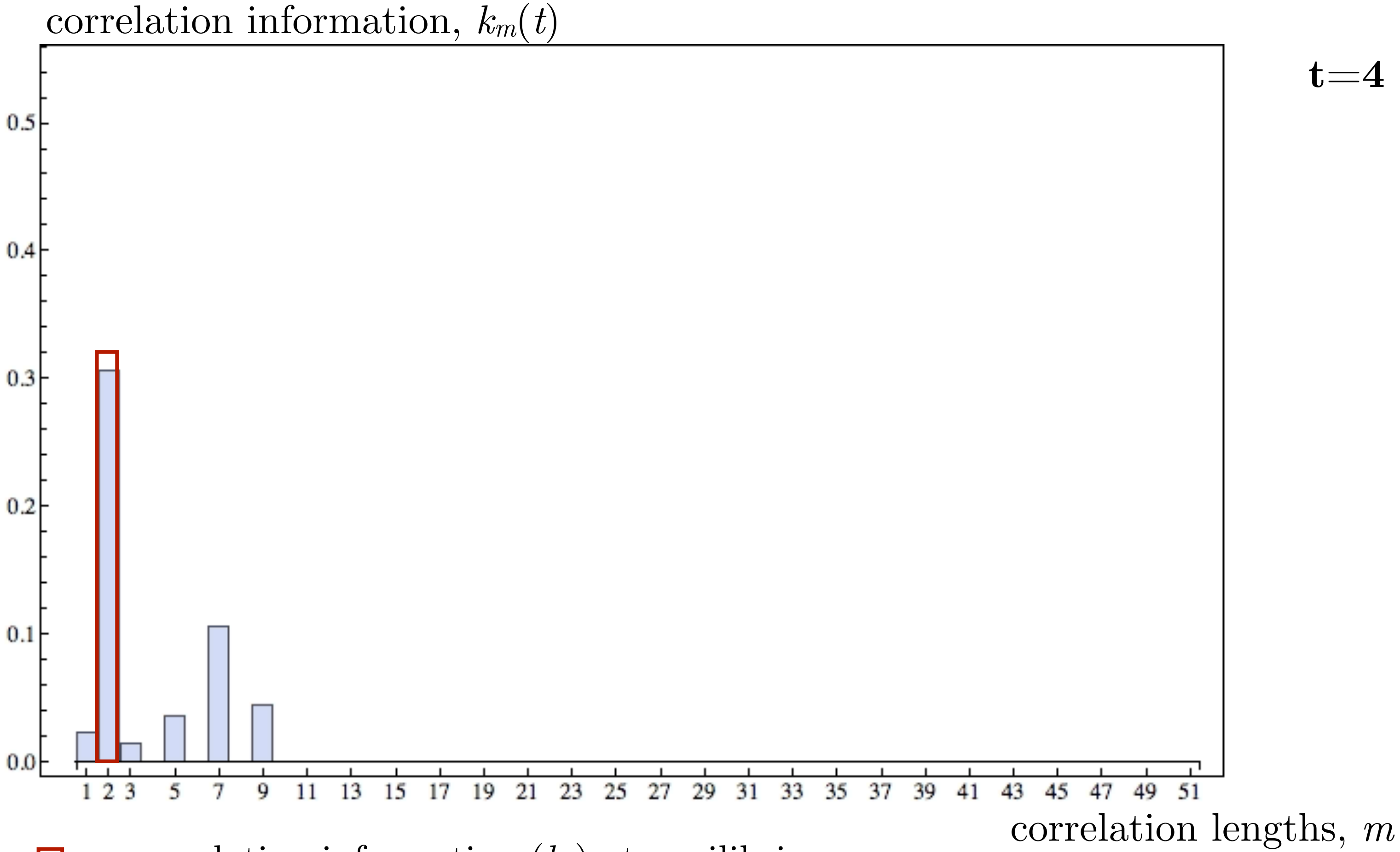

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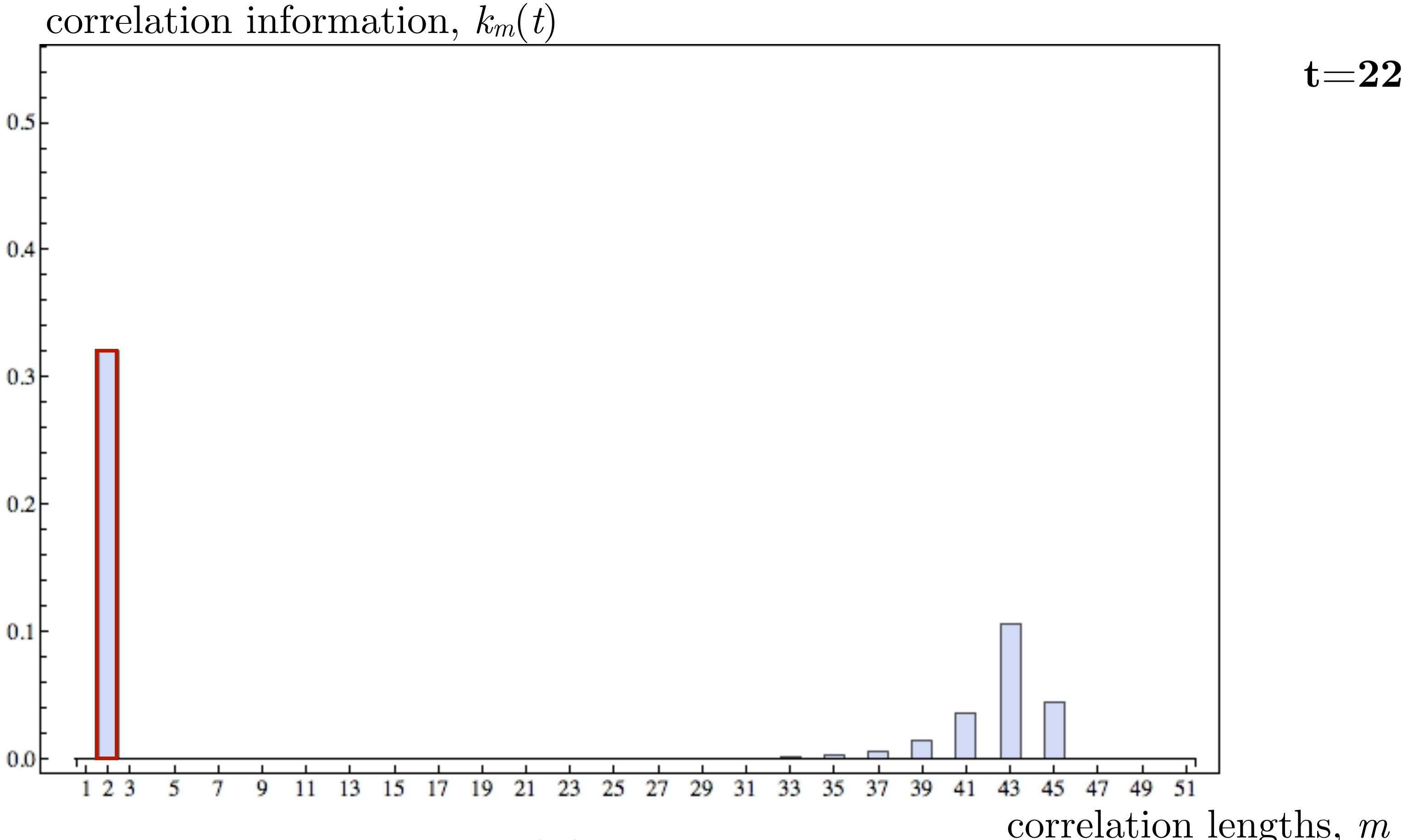

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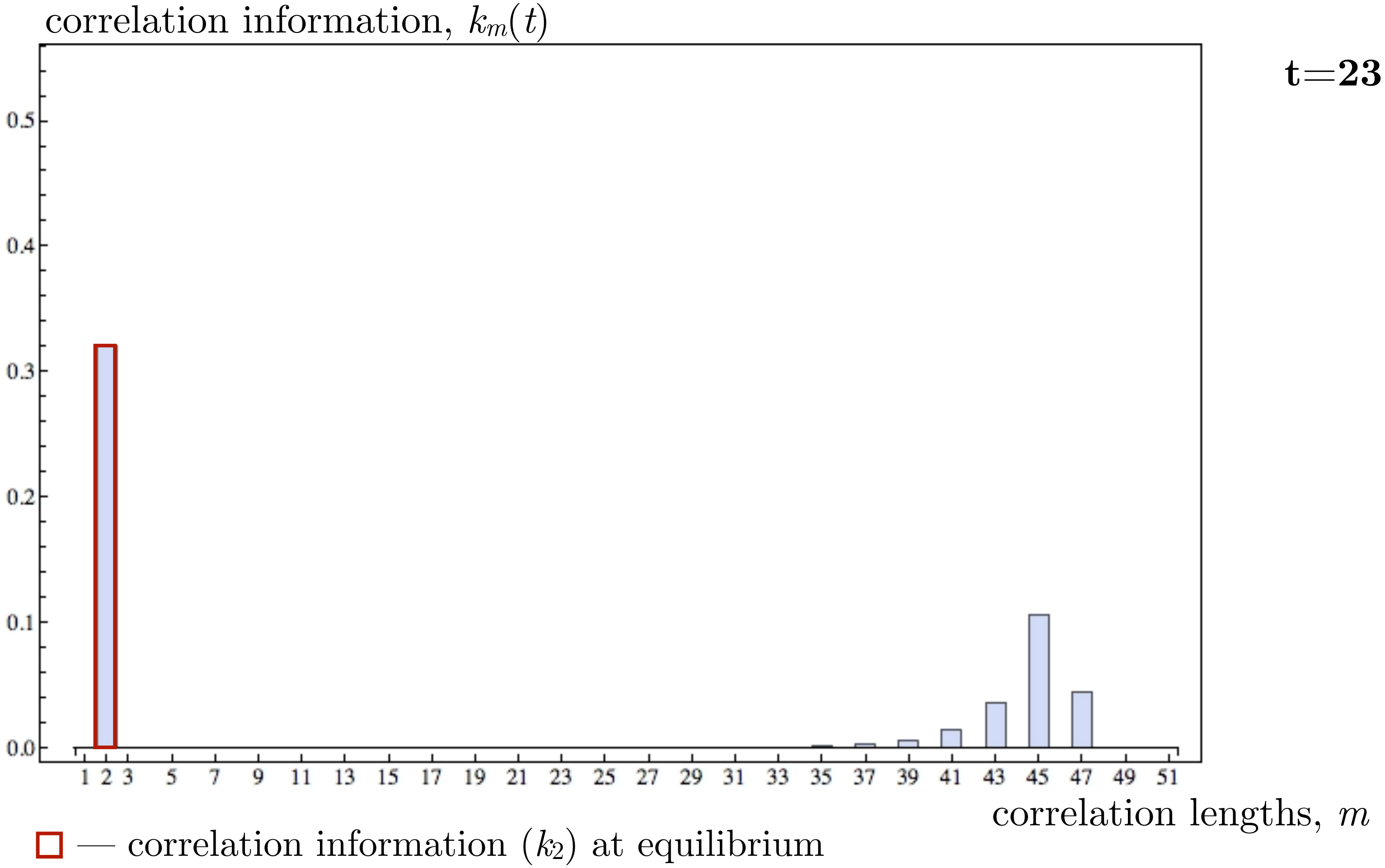

An increasing average correlation length

We prove that:

- There is no correlation information over even length blocks larger than 2, i.e., $k_{2m} = 0$ (for $2m > 2$).
- Correlation information is increasing step-wise in length for every time step: $k_{2m+1}(t+1) = k_{2m-1}(t)$.
- And we can finally prove that the correlation complexity η ($\sim$ average correlation length) increases linearly in time.

$$\eta = \sum_{m=1}^{\infty} m\, k_m(t) \;=\; (2t-1)\zeta + \log 2$$

The approach to equilibrium

In summary, this means that

- It appears as if the (the statistical mechanics) entropy increases to the corresponding equilibrium value.
- This is due to the fact that part of the initial order is transferred into correlation information of larger and larger distances.
- That long-range order loses its physical significance since it is no longer accessible (even though it is statistically present).
- The main contribution here is that this is mathematically shown in a conceptual model of a physically relevant system.