TIF 150: Information theory for complex systems

Time: March 22, 2019, afternoon.
Allowed material: Calculator (type approved according to Chalmers rules). “Beta”.
Examiner: Kristian Lindgren (0707574031)
Teacher: Susanne Pettersson (0706985466)

All answers and derivations must be clear and well motivated.
Grade limits: 25p for 3, 34p for 4, and 42p for 5.
ECTS grades: 25p for E, 28p for D, 34p for C, 38p for B, 42p for A.
The results will be available on April 12.

1. Balance measurements

You have 6 balls, and you know that 3 are of normal weight, while the remaining 3 are slightly heavier (having the same weight deviation), but you do not know which ones are deviating. Using as few balance measurements as possible (considering the worst case), you are expected to identify the deviating balls.

(a) According to information theory, what is the initial entropy of the 6-ball system? How much information would you get from ideal measurements using a balance (an ideal measurement is one that provides the most expected information)? What information does this provide about how many measurements you will need in order to solve the problem?

(3 p)

(b) Find the shortest sequence of measurements that (always) identifies the deviating balls. Determine the remaining uncertainty (entropy) after the successive measurement steps. You only need to consider the worst outcome (or one of them if there are several) in each measurement, i.e., the most probable measurement result.

(7 p)

2. An equilibrium spin system

Some material scientists are experimenting on making an infinitely long 1D atomic chain such that each atom binds to two other atoms. The atoms interact only with the ones they bind to (nearest neighbors) and the average interaction energy is u. The scientists want to use two types of atoms Np (Neptunium) and Pu (Plutonium), where the bonds Np-Np, Pu-Pu contribute with energy $\alpha > 0$ and Np-Pu, Pu-Np contribute with energy -2.

(a) Find the equilibrium distribution of the chain using the maximum entropy formalism. You may give the answer as a function of temperature instead of energy. **(6 p)**

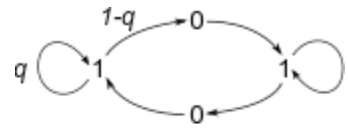
(b) Find the densities of Np and Pu and the pairwise probabilities at the limit of temperature T –> ∞, T –> 0 and describe what the chain looks like in the two cases. **(4 p)**

3. CA information.

Consider a one-dimensional cellular automaton given by elementary rule R75.

(a) Is this rule almost reversible or irreversible? **(1 p)**

Let the initial state be characterized by the following finite state automaton. Except for the explicit probabilities q and $1 - q$, assume that where multiple arcs leave the same node they have equal probability.



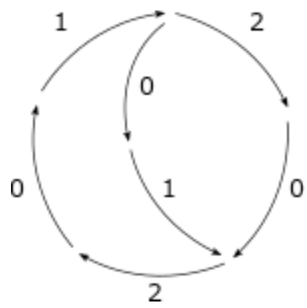
(b) What is the initial entropy density (time $t = 0$)? Hint: You may want to separately consider the cases $q = 1/2$ and $q \neq 1/2$. **(4 p)**

(c) Derive a finite state automaton that characterizes the CA state after one time step ($t = 1$). Like in the figure above, write the transition probabilities on the arcs leaving a node if they do not have equal probability. **(3 p)**

(d) What is the entropy density s at time $t = 1$? **(2 p)**

4. Correlation complexity.

Consider the symbol sequence generated by this hidden Markov model.



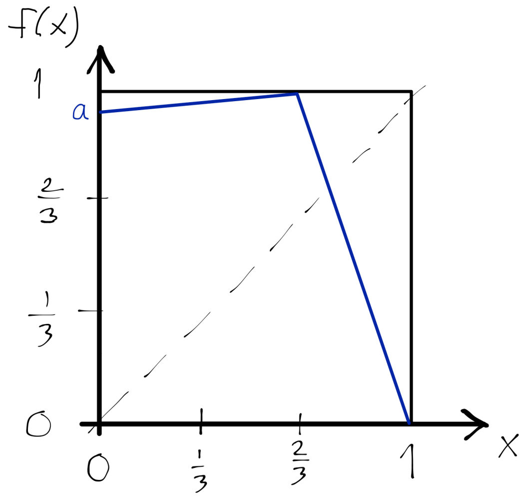
(a) How long correlations are there in the symbol sequence? In other words, what is the largest m for which $k_m > 0$? **(3 p)**

(b) What is the entropy of the symbol sequence? **(2 p)**

(c) What is the correlation complexity of the sequence? **(5 p)**

5. Chaos and information.

Let a piecewise linear map $f(x)$ be defined by the figure below, where $2/3 \le a \le 1$, and where the map is determined by $f(0) = a$, $f(2/3) = 1$, and $f(1) = 0$.



Consider the dynamical system

$$x_{t+1} = f(x_t) .$$

(a) Start with a at 1 and discuss how the dynamics changes when a is decreased. Determine whether there is a stable fixed point, stable periodic orbit, or chaos. Is there a critical value for a, for which there is a change in dynamical characteristics?

(b) Suppose now that $a = 2/3$:

- Determine the invariant measure that characterizes the chaotic behaviour, and calculate the Lyapunov exponent λ.
- Find a partition that is generating, and calculate the measure entropy from the symbolic dynamics.

(10 p)

Information theory for complex systems – useful equations

Basic quantities

$$I(p) = \log\frac{1}{p} \qquad S[P] = \sum_{i=1}^{n} p_i \log\frac{1}{p_i} \qquad K\left[P^{(0)};P\right] = \sum_{i=1}^{n} p_i \log\frac{p_i}{p_i^{(0)}}$$

Max entropy formalism (with $k+1$ constraints) using the Lagrangian L

$$L(p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_r,\mu) = S[P] + \sum_{k=1}^{r}\lambda_k\left(F_k - \sum_{i=1}^{n} p_i f_k(i)\right) + (\mu-1)\left(1-\sum_{i=1}^{n} p_i\right),$$

$$p_j = \exp\left(-\mu - \sum_{k=1}^{r}\lambda_k f_k(j)\right), \quad \mu(\lambda) = \ln\sum_{j=1}^{n}\exp\left(-\sum_k \lambda_k f_k(j)\right), \quad \frac{\partial\mu(\lambda)}{\partial\lambda_k} = -F_k$$

Symbol sequences

$$s = \lim_{n\to\infty}\sum_{x_1\ldots x_{n-1}} p(x_1\ldots x_{n-1})\sum_{x_n} p(x_n \mid x_1\ldots x_{n-1})\log\frac{1}{p(x_n \mid x_1\ldots x_{n-1})} =$$

$$= \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \Delta S_\infty = \lim_{n\to\infty}\frac{1}{n}S_n$$

$$k_1 = \log\nu - S_1, \qquad k_n = -S_n + 2S_{n-1} - S_{n-2} = -\Delta S_n + \Delta S_{n-1} = -\Delta^2 S_n \quad (n = 2,3,\ldots)$$

$$k_m = \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1})K[P^{(0)};P] = \tag{3.16}$$

$$= \sum_{x_1\ldots x_{m-1}} p(x_1\ldots x_{m-1})\sum_{x_m} p(x_m|x_1x_2\ldots x_{m-1})\log\frac{p(x_m|x_1x_2\ldots x_{m-1})}{p(x_m|x_2\ldots x_{m-1})},$$

$$S_{\text{tot}} = \log\nu = \sum_{m=1}^{\infty}k_m + s$$

$$\eta = \sum_{m=1}^{\infty}(m-1)k_m = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\sigma_m} p(\sigma_m)\sum_{\tau_n} p(\tau_n \mid \sigma_m)\log\frac{p(\tau_n \mid \sigma_m)}{p(\tau_n)} = \lim_{m\to\infty}(S_m - m\,s)$$

Geometric information theory

$$p(r;x) = \frac{1}{\sqrt{2\pi}\,r}\int_{-\infty}^{\infty} dw\, e^{-w^2/2r^2}\,p(x-w), \qquad \left(-r\frac{\partial}{\partial r} + r^2\frac{d^2}{dx^2}\right)p(r;x) = 0$$

$$p_{\text{Gaussian}}(r;x) = \frac{1}{\sqrt{2\pi}\sqrt{b^2+r^2}}\exp\left(-\frac{x^2}{2(b^2+r^2)}\right)$$

$$K[p_0;p] = \int dx\, p(x)\ln\frac{p(x)}{p_0(x)} = \int_0^{\infty}\frac{dr}{r}\int dx\, k(r,x), \qquad k(r,x) = r^2 p(r;x)\left(\frac{d}{dx}\ln p(r;x)\right)^2$$

$$d(r) = D_{\text{E}} - r\frac{\partial}{\partial r}\int d\mathbf{x}\, p(r;\mathbf{x})\ln\frac{1}{p(r;\mathbf{x})}$$

Chaos and information

$$\int dx \mu(x) \varphi(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \varphi(f^k(x(0))),$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln|f'(x(k))| = \int dx \mu(x) \ln|f'(x)|$$

$$h(\mu,\mathbf{A}) = \lim_{n\to\infty} \frac{1}{n} H(B^{(n)}) = \lim_{n\to\infty} \left(H(B^{(n+1)}) - H(B^{(n)})\right), \quad s_\mu = \lim_{diam(\mathbf{A})\to 0} h(\mu,\mathbf{A}), \quad s_\mu = \lambda$$